Math 501: Measure Theory Notes

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Last updated: November 15, 2024

Contents

※ Basics of Set Theory

1.1 Elementary Sets Operations

Definition 1.1. A **Set** is a well defined collection of items, things, objects, etc, and it is denoted by,

$$S = \{s_i\},$$

where the s_i are called the elements of S

Example 1.2. Following are the examples of sets.

1. $\mathbb{N} := \{0, 1, 2, ...\}$ (Natural Numbers)
2. $\mathbb{N}_+ := \{1, 2, 3, ...\}$ (Positive Natural Numbers)
3. $\mathbb{Z} := \{0, \pm1, \pm2, ...\}$ (Integers)
4. $\mathbb{Q} := \left\{ \frac{a}{b} \middle| \text{ a, b in } \mathbb{Z}; b \neq 0 \right\}$ (Rational Numbers)
5. $\mathbb{R}-$ Completion of $\mathbb{Q}$ with respect to metric $|x - y|$ (Real Numbers)

Similarly for $\mathbb{N}^d, \mathbb{Z}^d, \mathbb{Q}^d, \mathbb{R}^d$ as d-dimensional extensions.

Definition 1.3. Let $f : dom(f) = X \to range(f) \subseteq Y$ be a function. The for all $A \subseteq dom(f)$, we denote

$$f(A) := \{f(x)|x \in A\},$$

which is called as **image** of A under f. For all $B \subseteq range(f)$, we denote

$$f^{-1}(B) := \{x \in dom(f)|f(x) \in B\}$$

which is called as **inverse image** of B under f (or simply **pre-image** of B under f).
The function f is said to be **onto** or **surjective** if and only if f(X) = Y.
The function f is said to be **one-to-one** or **injective** if and only if for all $x_1, x_2 \in dom(f)$, we have $f(x_1) = f(x_2) \to x_1 = x_2$.

Example 1.4. Any sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n \in X$ may be viewed as a function

$$f := \mathbb{N} \to \text{ X via n} \in \mathbb{N} \to f(n) = x_n.$$

Definition 1.5. A set S is said to be **Countable** if and only if there exists a surjective mapping $f := \mathbb{N} \to S$. Otherwise, S is said to be **Uncountable**.
If the cardinality given as #(S) of such a countable set S is infinite, we say that S is **Countably Infinite**.
If the cardinality $\#(S) < +\infty$ of the set S is infinite, then S is **Countably finite**.
We denote it by,

$$\mathscr{P}(s) := \{X | X \subseteq S\}$$

$$\mathscr{P}(s) := (2^S)$$

and it is called as a **Power-set** of S.

Proposition 1.6. A Set S is countably infinite if and only if there exists a bijection $f : \mathbb{N} \to S$.

Proof. $\implies$
By Definition, we know that an Infinite set S is countably infinite if there is a bijection from the set of Natural numbers onto the set S.
$\impliedby$
If set S has a bijection on the set of natural numbers $\mathbb{N}$ then by the definition of the cardinality, we have that $|\mathbb{N}| = |S|$. Thus S is countably infinite. □

Example 1.7. The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countably infinite

$$f : \mathbb{N} \to \mathbb{Z} \text{ via n } \to f(n) = \begin{cases} \frac{n}{2}, & \text{n is even} \\ -\frac{n+1}{2}, & \text{n is odd} \end{cases}$$

However, $\mathbb{R}^d, 2^d, [0, 1], [a, b]$ are countable.

Claim. $2^{\mathbb{N}}$ is not countable

Proof. Suppose we assume the contrary. Let $2^{\mathbb{N}}$ be countable set.
Using the definition of the countable set, we know that there is a surjection of the set $2^{\mathbb{N}}$ to the set of Natural numbers.
Then we have a mapping of a function f as,

$$f : \mathbb{N} \to 2^{\mathbb{N}}$$

We define $A := \{n \in \mathbb{N} | n \neq f(n)\}$
We must recall that f is one-to-one, that is there exists some m $\in \mathbb{N}$ such that f(m) = A.

However, by the definition of A we have, $m \neq f(m)$. Here we have a contradiction. Thus, our assumption was incorrect.
We conclude that $2^{\mathbb{N}}$ is not countable. □

Claim. [0, 1] is uncountable.

Proof. We will prove this by the method of contradiction.
Suppose the set [0, 1] is countable. Then by definition, we can enumerate all the members of the set [0, 1] with natural numbers.
So,

$$[0, 1] = x_1, x_2, x_3, ..., \forall x \in [0, 1]$$

[0, 1] has a decimal representation given as,

$$x = 0.b_1b_2b_3b_4.........$$

Suppose there is an enumeration of all numbers, $x_1, x_2, x_3, ...$ in [0,1]

$$x_1 = 0.b_{11}b_{12}b_{13}..................b_{1n}....$$
$$x_2 = 0.b_{21}b_{22}b_{23}..................b_{2n}....$$
$$x_3 = 0.b_{31}b_{32}b_{33}..................b_{3n}....$$
$$.$$
$$.$$
$$.$$
$$x_n = 0.b_{n1}b_{n2}b_{n3}..................b_{nn}....$$

We can now construct a real number in [0, 1], which is not listed with the set of Natural numbers.
Let $y = 0.y_1y_2y_3..................y_n....$, such that we have,

$$y_1 \neq b_{11}$$
$$y_2 \neq b_{22}$$
$$y_3 \neq b_{33}$$
$$y_4 \neq b_{44}$$
$$.$$
$$.$$
$$.$$

$$y_n \neq b_{nn}$$

Here y is not equal to any of the number with two decimal representation, since $y_n \neq 0, 9$. Thus y and x_n differ in the n^{th} place, so $y_n \neq x_n$ for any $n \in \mathbb{N}$. Therefore, y is not included in the enumeration of [0,1] which is a contradiction to our statement. Hence, [0, 1] is uncountable. □

Definition 1.8. Let A, B be sets. Then.

1. A $\overset{def}{=}$ B iff A ⊆ B and B ⊆ A, $\iff (\forall x \in A, \implies x \in B$ and $\forall x \in B \implies x \in A)$
2. $\varnothing := \{\}$ **empty** or **void** set which contains no element.
3. A ⊊ B iff $A \subseteq$ B and $A \neq B$.

Definition 1.9. Let A, B ⊆ X be sets.
The **Union** of A with B is given by,

$$A \cup B := \{x \in X | x \in A \text{ or } x \in B\}$$

The **Intersection** of A and B is given by,

$$A \cap B := \{x \in X | x \in A \text{ and } x \in B\}$$

The **Difference** of A and B is given by,

$$A \backslash B := \{x \in X | x \in A \text{ and } x \neq B\}$$

The **Symmetric-Difference** of A and B is defined as,

$$A \triangle B := (A \cup B) \backslash (A \cap B)$$

Two sets A and B are said to be **Disjoint** if and only if,

$$A \cap B = \varnothing$$

Proposition 1.10. Distributive Laws

$$(i)(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(ii)(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(iii)(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$$
$$(iv)(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$$

Proof. (i) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
We will prove the above by the method of LHS contained in RHS and RHS contained in LHS.
First we will prove that, $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$ (1)
Let x $\in A \cap (B \cup C)$, then,

$$\begin{aligned} x \in A \cap (B \cup C) &\implies x \in A \text{ and } (x \in B \text{ or } x \in C)(\text{ Def. of Uni and Intersect}) \\ &\implies (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)(\text{Dist. law of logic}) \\ &\implies (x \in A \cap B) \text{ or } (x \in A \cap C)(\text{ Def of intersection}) \\ &\implies x \in (A \cap B) \cup (A \cap C)(\text{ Def of Unions.}) \end{aligned}$$

Now we will prove the other way. $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$ (2)
Let x $\in (A \cap C) \cup (B \cap C)$, then,

$$\begin{aligned} x \in (A \cap C) \cup (B \cap C) &\implies (x \in A \cap B) \text{ or } (x \in A \cap C)(\text{Def. of Uni and Inter}) \\ &\implies (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)(\text{Dist. law}) \\ &\implies x \in A \text{ and } (x \in B \text{ or } x \in C)(\text{ Def of Union}) \\ &\implies x \in A \cap (B \cup C)(\text{ Def of Intersect.}) \end{aligned}$$

From (1)and (2), we have that, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
Similarly one can show the problem (ii) as well. □

Proof. (iii) We need to prove that $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$

$$\begin{aligned} A \backslash B \cup C &= A \backslash (B \cup C) \\ &= A \cap (B \cup C)^c \\ &= A \cap (B^c \cap C^c) \\ &= (A \cap B^c) \cap (A \cap C^c) \\ &= (A \backslash B) \cap (A \backslash C) \end{aligned}$$

Similarly one can prove (iv) as well. □

Definition 1.11. Let A $\subseteq$ X be a set. The **complement** of A, denoted as A^c or $(\bar{A})$, is defined to be the set

$$A^c := X \backslash A; \quad (= \{x \in X : x \notin A\})$$

Proposition 1.12. $\forall$ sets A, B $\subseteq$ X,

$$(A^c)^c = A, A \cap A^c = \varnothing, A \cup A^c = X,$$

$$A \backslash B = A \cap B^c, A \subseteq B \iff B^c \subseteq A^c$$
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Proof. (i) To prove that $(A^c)^c$, we need to verify the two containments, $(A^c)^c \subseteq A$ and $A \subseteq (A^c)^c$.
We will begin by showing that $(A^c)^c \subseteq A$. Suppose that $x \in (A^c)^c$. By the definition of complement, this means that $x \neq (A^c)$.
But this says precisely, that x is not in A^c, which by the definition of the complement again, means exactly that x is in A. In other words, x $\in A$.
To show that $A \subseteq (A^c)^c$. Assume that $x \in A$.
By the definition of complement, this means that x is not in (A^c). In other words, $x \in A^c$ so that $x \in (A^c)^c$. □

Proof. (ii) Use the definition of Difference to show $A \backslash B = A \cap B^c$. □

Proof. (iii) and (iv) The main idea of these proofs is that negation changes "and" into "or" and vice-versa. Therefore, we only prove the first law and second one follows the same.
$\Longrightarrow$ Suppose x $\in (A \cap B)^c$. This means $x \notin (A \cap B)$. Notice that the negation of "$x \in A$ and $x \in B$" is equivalent to "$x \notin A$ or $x \notin B$". This implies that $x \in A^c$ or $x \in B^c$. In other words, $x \in A^c \cup B^c$.
$\Longleftarrow$ Suppose x $\in A^c \cup B^c$. This means $x \notin A$ or $x \notin B$. This is logically equivalent to the negation of "x $\in A$ and x $\in B$". In other words, it is equivalent to the negation of x $\in A \cap B$. We may conclude that $x \notin (A \cap B)$ that is, $x \in (A \cap B)^c$. □

1.2 Indicator Functions, Set Limits, Convergence of Sets and Monotonicity

Definition 1.13. Let A, X be sets with $A \subseteq$ X. The function $x \in X \to \mathbb{I}_A : X \to \{0, 1\}$ defined by,

$$\mathbb{I}_A(x) = \begin{cases} 1 : x \in A \\ 0 : x \notin A \end{cases}$$

is said to be the **Indicator function** of A or **Characteristic function** of A (relative to X).

Lemma 1.14. Let A, B $\subseteq$ X, Then,

1. $A \subseteq B \iff \mathbb{I}_A \leq \mathbb{I}_B$

Proof. For A, B $\in X$, $A \subseteq B \iff \mathbb{I}_A = \mathbb{I}_B$
B can be written as, $B = (A \cap B) \cup (A^c \cap B)$
So, if x $\in A \cap B, \mathbb{I}_A = \mathbb{I}_B = 1$
If x $\in A^c \cap B, \mathbb{I}_A = 0 < 1 = \mathbb{I}_B$ Hence, $\mathbb{I}_A \leq \mathbb{I}_B$ whenever, $A \subseteq B$. □

2. $\mathbb{I}_{A\cap B} = \mathbb{I}_A \cdot \mathbb{I}_B$

Proof. For A, B $\in X$, we need to show that $\mathbb{I}_{A\cap B} = \mathbb{I}_A \cdot \mathbb{I}_B$

$$\begin{aligned}\mathbb{I}_{A\cap B} = 1 &\iff x \in A \cap B \\ &\iff x \in A \text{ and } x \in B \\ &\iff \mathbb{I}_A \cdot \mathbb{I}_B = 1.\end{aligned}$$

Similarly, we can also show for the value 0.

$$\begin{aligned}\mathbb{I}_{A\cap B} = 0 &\iff x \in A \cap B \\ &\iff x \in A \text{ and } x \in B \\ &\iff \mathbb{I}_A \cdot \mathbb{I}_B = 0.\end{aligned}$$

Thus, $\mathbb{I}_{A\cap B} = \mathbb{I}_A \cdot \mathbb{I}_B$ □

3. $\mathbb{I}_{A\cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_{A\cap B}$

Proof. For any A, B $\in X$, We need to show that $\mathbb{I}_{A\cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_A \cdot \mathbb{I}_B$
Note that $A \cup B = (A \cup (A^c \cap B))$ and we can also write, B = $(A \cap B) \cup (A^c \cap B)$.

$$\mathbb{I}_B(x) = \mathbb{I}_{A\cap B}(x) + \mathbb{I}_{A^c\cap B}(x)$$

$$\mathbb{I}_{A^c\cap B}(x) = \mathbb{I}_B(x) - \mathbb{I}_{A\cap B}(x)$$

Now since $A \cup (A^c \cap B) = \varnothing$, we have from the above equation,

$$\begin{aligned}\mathbb{I}_{A\cup B}(x) = \mathbb{I}_{A\cup(A^c\cap B)}(x) &= \mathbb{I}_A(x) + \mathbb{I}_{A^c\cap B}(x) \\ &= \mathbb{I}_A(x) + \mathbb{I}_B(x) - \mathbb{I}_{A\cap B}(x)\end{aligned}$$

□

4. $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A = \mathbb{I}_X - \mathbb{I}_A$

Proof. For any A, $\in X$, we know that $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A$
Then, if x $\in A^c, \mathbb{I}_{A^c} = 1 = 1 - 0 = 1 - \mathbb{I}_A$ (Since $\mathbb{I}_A = 1$)
If x $\notin A^c$, which implies that $x \in A, \mathbb{I}_{A^c} = 0 = 1 - 1 = 1 - \mathbb{I}_A$ □

5. $\mathbb{I}_{A\backslash B} = \mathbb{I}_A(1 - \mathbb{I}_B)$

Proof. The set difference is given by, $A\backslash B = A \cap B^c$

$$\begin{aligned}\mathbb{I}_{A\backslash B} &= \mathbb{I}_{A\cap B^c}\\ &= \mathbb{I}_A \cdot \mathbb{I}_{B^c}\\ &= \mathbb{I}_A \cdot (1 - \mathbb{I}_B)\\ &= \mathbb{I}_A(1 - \mathbb{I}_B\end{aligned}$$

□

6. $\mathbb{I}_{A\triangle B} = |\mathbb{I}_A - \mathbb{I}_B|$

Proof. The symmetric difference is given as, $A\triangle B = (A\backslash B) \cup (B\backslash A)$

$$\begin{aligned}\mathbb{I}_{A\triangle B} &= \mathbb{I}_{((A\backslash B)\cup(B\backslash A))}\\ &= \mathbb{I}_{A\backslash B} + \mathbb{I}_{B\backslash A}\\ &\leq |\mathbb{I}_{A\backslash B}| + |\mathbb{I}_{B\backslash A}|\\ &= |\mathbb{I}_{A\backslash B} - \mathbb{I}_{B\backslash A}|\\ &= |\mathbb{I}_{A\backslash B} + \mathbb{I}_{A\cap B} - \mathbb{I}_{A\cap B} + \mathbb{I}_{B\backslash A}|\\ &= |\mathbb{I}_A - \mathbb{I}_B|\end{aligned}$$

□

Definition 1.15. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets $A_n \subseteq X$, Then,

$$\limsup_{n\to+\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \text{ } \textbf{(Limes Superior)},$$

and

$$\liminf_{n\to+\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \text{ } \textbf{(Limes Inferior)},$$

A sequence $(A_n)_{n\geq 1}$ is **convergent** $\iff \limsup_{n\to+\infty} A_n = \liminf_{n\to+\infty} A_n$
$(A_n)_{n\geq 1}$ is **increasing** iff $\forall n \in \mathbb{N}, A_n \subseteq A_{n+1}$.
Similarly, **Decreasing, Strictly Increasing, and Strictly Decreasing** are defined as expected.
$(A_n)_{n\geq 1}$ is **Monotone** iff it is increasing or decreasing.

Remark 1.16. Trivially always,

$$\liminf_{n\to+\infty} A_n \subseteq \limsup_{n\to+\infty} A_n$$

Theorem 1.17. Every monotone sequence $(A_n)_{n\geq 1}$ with $A_n \subseteq X$ converges. Moreover, we have,

$$\lim_{n\to+\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

if (A_n) is increasing and,

$$\lim_{n\to+\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

if (A_n) is decreasing.

Proof. Suppose $(A_n)_{n\geq 1}$ is increasing. Then,

$$\liminf_{n\to+\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n \supseteq \limsup_{n\to+\infty} A_n \supseteq \liminf_{n\to+\infty} A_n$$

$$\Longrightarrow \liminf_{n\to+\infty} A_n = \limsup_{n\to+\infty} A_n$$

Suppose $(A_n)_{n\geq 1}$ is decreasing. Then,

$$\liminf_{n\to+\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_{n\to+\infty} A_n$$

□

Remark 1.18. Note that,

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \liminf_{n\to+\infty} A_n = \{x \in X | x \in A_n, \text{ for infinitely many n } \in \mathbb{N}\},$$

hence,

$$\limsup_{n\to+\infty} A_n \{x \in X | \exists n_0 \in \mathbb{N} : \forall n \geq n_0(x), x \in A_n\}$$

For $(A_n)_{n\geq 1}$ increasing,

$$A_n \uparrow A_\infty.$$

For $(A_n)_{n\geq 1}$ decreasing,

$$A_n \downarrow A_\infty.$$

Similarly, $f_n \uparrow f_\infty$ and $f_n \downarrow f_\infty$
For the real valued sequences, $(a_n)_{n\in\mathbb{N}}$, we have,

$$\liminf_{n\to+\infty} a_n = \liminf_{n\to+\infty} \{a_k : k \geq n\} = \sup_k \inf \{a_n : n \geq k\},$$

$$\limsup_{n\to+\infty} a_n = \limsup_{n\to+\infty} \{a_k : k \geq n\} = \inf_k \sup \{a_n : n \geq k\}$$

Lemma 1.19. $\forall A_n, B_n, C \subseteq X$

1. $(\limsup_{n\to+\infty} A_n)^c = \liminf_{n\to+\infty} A_n^c$

Proof.

$$
\begin{aligned}
(\limsup_{n\to+\infty} A_n)^c &= (\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k)^c \\
&= \bigcup_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k)^c \\
&= \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^c \\
&= \liminf_{n\to+\infty} A_k^c
\end{aligned}
$$

□

2. $\mathbb{I}_{\liminf_{n\to+\infty} A_n} = \liminf_{n\to+\infty} \mathbb{I}_{A_n}$
   $\mathbb{I}_{\limsup_{n\to+\infty} A_n} = \limsup_{n\to+\infty} \mathbb{I}_{A_n}$

   Proof. □

3. $A_n \to C \iff \mathbb{I}_{A_n} \to \mathbb{I}_C$

   Proof. □

4. $\liminf_{n\to+\infty} A_n \cup \limsup_{n\to+\infty} B_n \subseteq \limsup_{n\to+\infty}(A_n \cup B_n)$

   Proof. □

5. $\limsup_{n\to+\infty} A_n \backslash \liminf_{n\to+\infty} A_n = \limsup_{n\to+\infty}(A_n \triangle A_{n+1})$

   Proof. □

6. $\exists \triangle_{n=1}^{\infty} A_n = A_1 \triangle A_2 \triangle ... \iff \lim_{n\to+\infty} A_n = \varnothing$

   Proof. □

Remark 1.20. $(\mathscr{P}(X), \triangle)$ forms an Abelian group (i.e. commutative and associative w.r.t $\triangle$ operations).

1.3 Family of Sets

Definition 1.21. A family of sets is a non empty set F whose elements are sets by themselves, denoted by

$$F = \{A_i | i \in I\} or \{A_i\}_{i \in I},$$

where I is called the Index set or i Indices

Remark 1.22. Set operation are usually defined on F, for example, Unions

$$\underset{i \in I}{\cup} A_i = \{x \in X | \exists i \in I \text{ such that } x \in X\}$$

and Intersections,

$$\underset{i \in I}{\cap} A_i = \{x \in X | \forall i \in I \text{ such that } x \in X\}$$

Theorem 1.23. Let A_i and B be the subsets of X. Then,

1. $(\underset{i \in I}{\cup} A_i) \cap B = \underset{i \in I}{\cup} (A_i \cap B)$

   Proof. Let x $\in \underset{i \in I}{\cup} A_i$ and $x \in B$, then,

   $$x \in (\underset{i \in I}{\cup} A_i) \cap B \iff x \in A_i, \forall i \in I \text{ and } x \in B$$

   This implies that,

   $$x \in \underset{i \in I}{\cup} (A_i \cap B)$$

   Conversely, if

   $$x \in \underset{i \in I}{(\cup)} (A_i \cap B) \iff x \in A_i \cap B, \forall i \in I$$

   This implies that

   $$x \in A_i \text{ and } x \in B$$

   then we have,

   $$x \in (\underset{i \in I}{\cup} A_i) \cap B$$

   Hence, we proved that

   $$(\underset{i \in I}{\cup} A_i) \cap B = \underset{i \in I}{\cup} (A_i \cap B)$$

   □

2. $(\underset{i \in I}{\cap} A_i) \cup B = \underset{i \in I}{\cap} (A_i \cup B)$ (Distributive Laws)

Proof. If x $\in B \cup (\bigcap_{i\in I} A_i)$, then $x \in B$ or $x \in \bigcap_{i\in I} A_i$.
This implies that $x \in A_i \forall i \in I$
Thus, $x \in B \cup A_i, \forall i \in I$ □

3. $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$ (De Morgan's Laws)

   Proof. □

4. $(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c$ (De Morgan's Laws)

   Proof. □

Definition 1.24. Let $f : X \to Y$ be one-to-one and onto. Then $f^{-1} : Y \to X$ defined by

$$f^{-1}(y) = x \iff f(x) = y$$

is called the inverse of f. Given two functions $f : X \to Y$ and $g : Y \to Z$. Then $x \mapsto (g \circ f)(x) = g(f(x))$ is called the composition $g \circ f : X \to Z$.

Theorem 1.25. On relationship between Images and Inverse Images.
Assume $(a_i)_{i\in I}$ is a family of subsets of X, $(B_i)_{i\in I}$ is a family of subsets of Y. Function $f : X \to Y$, Then we have the following

1.4 Cartesian Product, Relation and Ordered Sets

1.5 Uncountable, Countable Sets, Cardinality as Subadditive Measure

1.6 Semiring, Ring, Algebra, $\sigma-$Algebra, and Borel Sets

1.7 Dynkin Systems and Minimal-Generated Systems

※ Metric Spaces

2.1 Basic concepts of Metric Spaces

References